Planning Bias: Planning as a Source of Sampling Bias

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Abstract

Many data-driven planning methods are trained on data generated by planners. It is well known that many statistical learning methods are sensitive to sampling bias, and yet there has been little or no attention to planning as a sampling method and its role in introducing sampling bias into planner-generated training data. Recently, it has been demonstrated that A*, in the presence of problems with variable heuristic error, prefers some solutions over other equally cost-optimal solutions. But, as we discuss in this paper, mitigation may not be as simple as resolving arbitrary tie-breaking by sampling from ties uniformly at random. In this paper, we formalize an intuition of *planning bias*. We focus on problems which require a single solution. Diverse planning only complicates the problem by generalizing it to bias in the set of sets; we show how it is subject to bias in the single solution. We make some useful observations about deterministic algorithms in contrast to non-deterministic algorithms. We explain how information entropy may be a good way to measure planning bias, and discuss some issues in evaluating practical approaches to measurement. We address the intuition that uniform random tie-breaking should mitigate bias, and sketch a novel approach to constructing an appropriate random distribution for duplicate detection during forward search for unbiased A*. Finally, we suggest directions for future work.

Introduction

The presence of planning bias has implications for a number of applications of automated planning, wherever planner-generated data are used as input to statistical models. Applications include training a neural network to iteratively improve a heuristic (Arfaee, Zilles, and Holte 2010; Ernandes and Gori 2004). Planner-generated examples may be used for automatic (Callanan et al. 2022; Mordoch et al. 2023) or assisted model elicitation (Feblowitz et al. 2021; Sohrabi et al. 2016). It has also been hypothesized that plan validation may benefit from statistical analysis of related plans for more robust validation via simulation (Benton et al. 2018).

Though it is generally well-understood that planning models may include biases—"All models are wrong, some are useful." (George Box)—a less understood source of bias is that which is introduced by the process of plan synthesis. Recently, it has been shown that A* favors some solutions among other equally cost-optimal solutions when heuristic error varies (Paredes 2023). However much work is still needed.

In this paper, we invite the planning community to look at planning as a source of sampling bias. Toward this end, we introduce the idea of *planning bias*. We believe that up until now data-driven planning and scheduling has assumed that planners are somehow a neutral source of examples. For some planning technology, that may indeed be the case. A brute force search may be a good sampling methodology for robust data-driven planning, assuming a planning model that we believe in. Faster methods, however, like A*, for example, are not necessarily unbiased sampling technology (Paredes 2023). We propose that robust data-driven planning and scheduling will require a much better understanding of *planning bias*. In the worst case, there may be an unavoidable trade-off between planning speed and planning bias. The topic has applications outside of planning such as in scheduling, constraint satisfaction, combinatorial design problems, and reinforcement learning (Trinh et al. 2024) and (Sartoretti et al. 2019). Below we attempt to formalize our intuition of planning bias, identify some major challenges, and present some initial results to stimulate discussion.

Preliminaries

We will use satisficing planning to develop some key ideas toward an intuition of *planning bias*.

Definition 1 (A Planning Task)**.** Let $M = \langle V, O \rangle$ be a set of variables and planning operators. Let I be an initial state consisting of a complete assignment to the variables. Let G be a set of goal states implicitly defined by an incomplete assignment to the variables. An operator is a function which transforms one state to another. Let $\Pi = \langle M, I, G \rangle$ be a planning problem instance. A solution s is a sequence of operators which transforms the initial state to a goal state.

A critical idea in the proceeding discussion is that of the solution space of the particular planning instance.

Definition 2 (Solution Space of a Planning Task)**.** Let $S(\Pi)$ be the space of solutions to Π. A plan is a solution s if and only if it is $\in S(\Pi)$. (To be clear, a sequence of operators which does not successfully transform the initial state into a goal state may never be an element of $S(\Pi)$). Thus $S(\Pi)$ contains all solutions s to Π and no invalid plans. This will be an important distinction for our purposes.

We can use this understanding of the solution space of a task to develop an idea of the probability of a solution, which we would like to be able to compare between algorithms.

Definition 3 (Probability Distribution Over the Solution Set)**.** Let $P_{\Pi}(s)$ be the probability of solution s to Π. Let $P_{A_i,\Pi}(s)$ be the probability that algorithm A_i generates solution s to Π. Let $P_{A_q i,\Pi}$ be the probability distribution of overall solutions s $\in S(\pi)$.

A useful idea is the algorithm that outputs any valid solution with equal probability:

Definition 4 (Uniform Distribution Over the Solution Set)**.** Let $P_{A_0,\Pi}(s) = \frac{1}{|S(\Pi)|}$, for all s. Then $P_{A_0,\Pi}$ is a uniform distribution over a planning task's solution space.

We can now measure the distance $\Delta(P_{A_i,\Pi}, P_{A_j,\Pi}) \Rightarrow \mathbb{R}_{\geq 0}$ between the uniform distribution over solutions and the distribution of solutions produced by planning algorithms. Different distance measures over distributions will give different answers, but this is not important for the development of our definition of bias in the next section.

A Formal Definition of Planning Bias

Our intuitive definition of algorithm bias is: if an algorithm does not produce valid solutions to a planning problem instance with uniform probability, it is biased. We formalize this as follows:

Definition 5 (Planning Bias)**.** Let A_p be a planning algorithm. $P_{A_0,\Pi}$ is a uniform distribution over a planning task's solution space (Definition 4). Then A_p is biased if $\Delta(P_{A_0,\Pi}, P_{A_p,\Pi}) > 0$.

Some useful results will help us more precisely describe our intuition of what planning bias looks like.

Theorem 1. Let A_i be a deterministic planning algorithm. Then A_i is biased if and only if $|S(\Pi)| > 1$.

Proof. A deterministic algorithm A_i returns the same solution s_i given the same inputs; equivalently, $P_{A_i,\Pi}(s_i) = 1$ for some s $\in$ S(Π) and $P_{A_i,\Pi}(s_j) = 0$, where $s_i \neq s_j$. Therefore, $|S(\Pi)| > 1 \Rightarrow \Delta(P_{A_0,\Pi}, P_{A_i,\Pi}) > 0$. Otherwise, $\Delta(P_{A_0,\Pi}, P_{A_i,\Pi}) = 0$. □

Theorem 2. Let A_e be the (EXPTIME)[1] algorithm that enumerates all solutions in S(Π) and samples from among them uniform at random. Then A_e is unbiased.

Proof. Trivial. □

Algorithm A_ϵ provides a theoretical upper bound on planning bias. We know such an algorithm is intractable given existing theoretical results. We defer readers to our sister paper (Frank et al. 2024) for a detailed discussion of the complexity of A_ϵ. The proof follows from the result that planning itself is PSPACE-complete except under extreme restrictions (Bylander 1994). If, in general, finding one solution is intractable, then, in general, finding all solutions is intractable. This result encourages us to turn our attention to the space of algorithms which work in practice. However, as an upper bound on planning bias, A_ϵ demonstrates the best we can hope for is indeed an unbiased algorithm. Where there is more work to be done is in answering the question, under what conditions may practical planning methods approach this ideal?

Theorem 3. Let A_i be a deterministic algorithm and A_j be a non-deterministic algorithm. $P_{A_0,\Pi}$ is a uniform distribution over a planning task's solution space (Definition 4). Then $\Delta(P_{A_0,\Pi}, P_{A_i,\Pi}) \geq \Delta(P_{A_0,\Pi}, P_{A_j,\Pi})$.

Proof. Trivial because any non-deterministic algorithm must return at least one plan. □

Deterministic algorithms must often make some arbitrary choices (e.g. tie-breaking). Non-deterministic algorithms make up a non-trivial class to consider, which includes many superficially deterministic algorithms. In practice, we might resolve random tie-breaking with a random seed to force determinism. In some cases, we hope through some judicious use of tie-breakers we can gain efficiency (Asai and Fukunaga 2016; Corrêa, Pereira, and Ritt 2018; Heusner, Keller, and Helmert 2018; Ferber et al. 2022).

Of non-deterministic algorithms, we should ask under what conditions they perform more like A_e (Theorem 2). We propose there are several challenges to answering this question. Some of them we go into more detail about below. For instance, there are challenges in practically computing bias without knowledge of the set of all solutions. How we might take advantage of existing ideas from information entropy? Such as addressing the intuition that all non-deterministic algorithms are unbiased at least in the long run.

What Planning Bias is Not That is not to say that planner-generated data cannot be biased due to biases in the model. A domain model that omits right turns will never output a solution containing right turns. The above discussion assumes a model we believe in. In other words, we are confident that the model is as accurate as it is ever going to get. Given that "all models are wrong but some are useful (George Box)," we assume a maximally useful model. Model elicitation is an important but different challenge.

It is tempting to think the proper approach to mitigating planning bias is to design the appropriate diversity metric. Diverse planning takes as input a planning task, a diversity metric, and an integer k, and returns a size-k set of solutions which conforms to the diversity metric. Diverse planning, however, only complicates the problem. We can easily generalize our definition of bias to diverse planning, where the solution to a diverse planning problem is not a single plan but a set of plans. The set of all solutions to the diverse planning problem is a set of sets. A diverse planning algorithm that will never output a particular set, a solution $s_{D,i}$, within the set of sets, all solutions to the diverse planning task, $S(\Pi_D)$, is biased.

[1]The algorithm is EXPTIME and not EXPSPACE because we can count the number of solutions without saving them, decide which should be returned, then generate them again and count solutions, returning the proper one when it is generated.

Some Thoughts on Estimating Bias

The previous section provides some notation which helps us think about bias but does not provide much guidance on how to practically identify it. We may be able to assume that most deterministic algorithms are biased. Per Theorem 1, a deterministic algorithm is biased if $|S(\Pi)| > 1$. If we knew the size of the solution space then we could easily determine the existence of planning bias. We may be able to approximate or infer the size of the solution space from features of the domain. For example, if a minimal cost solution is composed of more than one communitive action then there must be at least one other minimal cost solution. Many more restrictions on the domain model may be necessary to ensure the size of the solution set is at most one. It is hard to imagine planning could be very difficult under these kinds of restrictions. Fortunately, it is not necessary to know exactly the size of the solution space in the case of a deterministic algorithm.

Unfortunately, we are more interested in non-deterministic algorithms. It may be difficult to be sure of the existence of planning bias, as we have defined it. One way to identify if a non-deterministic algorithm is biased if we could analytically prove that algorithm A can never output some $s \in S(\Pi)$. If the probability of some solution via A is 0, then $\Delta(P_{A_0,\Pi}, P_{A,\Pi}) \neq 0$. In the absence of such a proof, experimental techniques may still be unsatisfying. We would need to find a solution that algorithm A will not produce. To experimentally show that some solution can never be output is undecidable.

Even if a planner can find all plans then we still cannot be confident it is not biased without additional work. Our definition of planning bias claims that any divergence from a uniform distribution over $S(\Pi)$ indicates bias, i.e. $\Delta(P_{A_i,\Pi}, P_{A_0,\Pi}) > 0$. A solution which is over-represented should indicate the existence of planning bias. Proving this empirically with any confidence will require appropriate tools.

Our intuition is to look to the *entropy* of $P_{A_i,\Pi}$. Recall that $P_{A_0,\Pi}$ represents the uniform distribution over $S(\Pi)$. Recall entropy is a measure of the uncertainty of the random variable. Let X be a discrete random variable with alphabet $\mathcal{X}$ and probability mass function $p(x) = Pr(X = x), x \in \mathcal{X}$.

$$H(X) = -\sum_{x \in \mathbf{X}} p(x) log(p(x)) \tag{1}$$

Entropy $H(X)$ is maximized when the probability distribution $p(x)$ is uniform. For our purposes, we claim that entropy would be maximized when A_i is unbiased, i.e. $\Delta(P_{A_0,\Pi}, P_{A_i,\Pi}) = 0$.

Definition 6 (Entropy of a planning algorithm)**.** Let S be a discrete random variable with alphabet $S(\Pi)$, the solution space of a particular planning task, and probability mass function $p(s) = Pr(S = s), s \in S(\Pi)$.

$$H(S) = -\sum_{s \in S(\Pi)} p(s) log(p(s)) \tag{2}$$

The theory of entropy provides some other useful properties for describing planning bias. Entropy is a convex function of the probability mass and monotonically decreases as the probability of an event increases, allowing us to compare whether an algorithm is more or less biased. When entropy equals zero, there is no uncertainty in the random variable; therefore, A_i is unbiased. Additionally, relative entropy between two random variables permits some level of comparison among algorithms. Relative entropy is zero if and only if two probability masses are equivalent, e.g. $P_{A_i,\Pi} = P_{A_j,\Pi}$.

Practically, however, it can be difficult to efficiently compute a good estimate of entropy, when the true distribution is unknown. There are many different ways to estimate entropy. Some of the most well-known include the naive estimator.

Definition 7 (A naive estimator as described in (Valiant and Valiant 2017))**.** Let S be a multiset representing a sample of solutions of size n. A sample could be generated by any non-deterministic planning algorithm by running it n times. Let F represent the fingerprint of S. A fingerprint is a vector whose ith component, F_i, is the number of elements that occur exactly i times in S. Then the entropy of $P_{A_i,\Pi}$ can be estimated from a sample of solutions as follows:

$$H^{naive}(F) := -\sum_i n_i \frac{i}{n} |log \frac{i}{n}| \tag{3}$$

One of the limitations of the naive estimator is in the number of samples it requires to get a good estimate. Recent work has shown that entropy can be estimated without *a priori* assumptions about the target distribution in only a sublinear-size sample, $O(k/logk)$, where k is the number of distinct elements in the target distribution (Valiant and Valiant 2017).

Randomized Approaches to Deterministic Algorithms

Since deterministic algorithms are almost certainly biased, we could replace any arbitrary choices in any deterministic algorithm with a random choice. But it may not be obvious what distribution to use in a randomization strategy that will ensure bias-free planning. Consider an example.

It was recently discovered that A* as originally defined with FIFO duplicate detection in (Hart, Nilsson, and Raphael 1968) may some equally cost-optimal solutions over others under realistic conditions (Paredes 2023). We show below that we cannot just replace FIFO duplicate detection with uniform random duplicate selection strategy; and we sketch an explanation of why.

In conventional A* (Hart, Nilsson, and Raphael 1968), search proceeds by expanding the node from open with the minimum f value until a goal node is reached. When the heuristic is admissible, the path it finds is guaranteed to be an optimal cost path. When the heuristic is also consistent, a closed list can improve time-efficiency. In this case, we can be sure, during search, when search expands a state for the first time it has found a minimal cost path to that state. First-in-first-out (FIFO) duplicate detection preserves only the first minimal cost path encountered during search.

The problem with FIFO duplicate selection is that the minimum cost path that is preserved for solution recovery

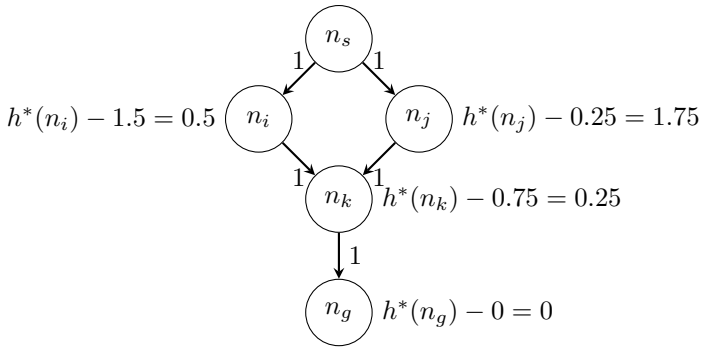

Figure 1: Adapted from (Paredes, 2023). The subgraph G_s of the state space under a admissible heuristic h. In this example, the true cost-to-go $h^*(n_i)$ and $h^*(n_j)$ are identical, however estimates $h(n_i) < h(n_j)$. The effect on solution recovery is that A* will always prefer the path through n_i. Randomized tie-breaking has no effect.

is influenced by the character of heuristic error in the problem. The result is a bias toward paths with poorer heuristic estimates.

Consider the search space illustrated in Figure 1. There are two minimal length solutions from the n_s to n_g, passing through n_k. Let solution s_i be the path through n_k via n_i, and solution s_j be the path through n_k via n_j. In this example, although the true cost-to-go $h^*(n_i)$ and $h^*(n_j)$ are identical, estimates $h(n_i) \neq h(n_j)$. This could happen for many reasons. Consider the effect on solution recovery.

Although states n_i and n_j both fall along minimal cost paths to n_g, A* will always prefer the path through n_i for solution recovery. Expansion will proceed as follows: $< n_s, n_i, n_k, n_j, n_g >$. Because A* will expand every node on open with f < f*, it will expand n_j and generate n_k a second time. But n_k will be discarded this time. A minimal cost path to n_k has already been found. And the solution s_j will never be recoverable.

One might think that this situation could easily be resolved with a random duplicate selection. The challenge here is in what distribution to use and how to compute it. Consider the naive approach: During duplicate detection, choose between the first node to close and the newly generated duplicate uniformly at random. More formally, let the incumbent be the first node closed. During duplicate detection, choose between the incumbent and the newly generated node with probability 0.5 and 0.5. Unfortunately, this approach selects the most recently generated duplicate.

The sequence of decisions, to keep the incumbent node or replace it with the newly discovered duplicate, form a Markov chain M. Each node in M represents a decision to keep the incumbent or replace it with the newly discovered duplicate. A directed edge from node A and B in M with label p denotes the probability of transition from A to B, the probability that the incumbent survives or the candidate replaces it. The probability that the first node put on closed survives to termination is the joint probability of surviving all previous decision points.

Our first intuition is to adjust the weight of the edges between decision points in proportion to the number of duplicates discovered so far. For example, when the first duplicate is discovered, the number of duplicates discovered so far is 2, therefore the probability that the incumbent survives or the candidate survives are weighted equally 0.5 and 0.5. When the second duplicate is discovered, the number of duplicates discovered so far is 3, and the probability that the incumbent survives is weighted 0.66 and 0.33. But this does not quite work because search will not discover all duplicates before termination. Each decision point effectively prunes the subtree rooted at the losing duplicate search node. This saves time in conventional A*. These efficiency gains come at the expense of bias-free search. It may take some careful bookkeeping to compute the number of duplicates for each decision point for the appropriate distribution to preserve the theoretical efficiency gains of conventional duplicate detection for A*.

Randomization even with the appropriate probability distribution may not be able to mitigate bias everywhere. Distance-guided heuristics for example in may be intrinsically biased toward shorter plans, their efficiency gains coming from an implicit restriction on the planning problem, making them more difficult to correct.

Discussion

Initial results in Paredes (2023) suggest that planning bias could be a bigger problem in generalizing methods using planner-generated data to real-world problems than we had realized. In this paper, we have defined planning bias more precisely to make some important distinctions from existing work in diverse planning and to explain our intuition that there may be a useful relationship between planning bias and information entropy for practically computing planning bias.

One of the limitations of our definition of planning bias is it assumes a finite solution set. A more general definition would allow for an infinite solution set, such as created when allowing any number of loops in the plan. The idea of a uniform distribution over an infinite (countable) set however is poorly defined. Further, the existence of a uniform distribution over the solution set plays a critical role in our results. Future work may develop a more general definition of planning bias and set of theoretical tools.

Practical tools to measure bias might generate a sample set of plans from a "time-limited planner" and measure entropy. Future work should compare the entropy estimate and bias estimates in an empirical study. We hope to show whether entropy works as a surrogate measure for bias in practice. For example, increasing the amount of time given to the planner (allowing us to find more plans) may change entropy for varying domains. Set-up may forbid finding the same plan twice.

We hope to investigate the landscape of bias in plan optimization algorithms further, especially those tackling problems in higher complexity classes (e.g., EXPTIME), to determine whether they exhibit bias both theoretically and empirically. For such complex problems, practitioners often use approximation algorithms and we hope to also examine how approximation approaches may express their (likely) biases.

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